Equations to Calculate Characteristic Frequencies of Multiple Chamber Aligned in Parallel Cavity Resonator (MCAP-CR)

Shigeru Suzuki

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1. Preface

It is necessary to solve the equations of motion, in order to estimate characteristic frequencies of Multiple Chamber Aligned in Parallel Cavity Resonator (MCAP-CR). I explain the equations of motion of MCAP-CR and suggested solution of the equation in this article.

Set of equations of motions is well arranged; however, it is not easy to solve the equations. Suggested solution will be explained in Appendix-B.

2. Physical Model of MCAP-CR

MCAP-CR consists of main chamber, sub-chambers, and ducts. Speaker unit(s) is installed in the main chamber, and sub-chambers are connected to main chamber through ducts. One or more sub-chamber has an open-to-air duct. Number of sub-chambers (let us define as N) must be two or greater. Fig. 1 shows illustration of an MCAP-CR that has four sub-chambers.

The chamber in the middle of Fig. 1 is the main chamber where a speaker unit is installed. Four sub-chambers are connected to the main chamber through ducts. All chambers have an open-to-air duct. Each chamber acts as air spring, and air in each duct acts as mass. In this case this system configures 8 degrees of freedom problem, because there are 8 masses.

One advantage of this system is that it could have twice as many characteristic frequencies of number of sub-chambers. Some of open-to-air ducts may be plugged in order to change characteristic frequencies.

Number of sub-chambers may be two or more and theoretically maximum number is infinity; practical limit will be approximately eight

[1] Original name of the system was MCAPSS (MCAP Speaker System).

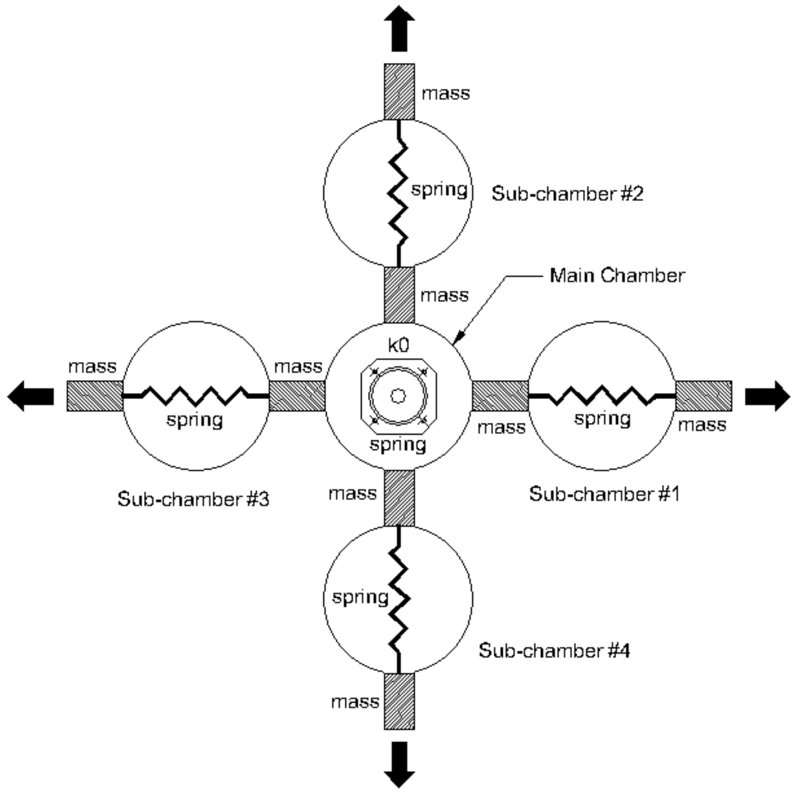

Fig.1 Schematic of MCAP-CR (N=4)

3. Basic Equations of Physical Model for MCAP-CR

We could say that MCAP-CR is a kind of bass-reflex system that has multiple characteristic frequency. It means basic equations can be extended from bass-reflex equations.

I begin with bass-reflex equations and extend the equations to MCAP-CR, because governing equations of MCAP-CR are more complex.

Equations of Single Bass-Reflex System

Single Bass-Reflex system is a cavity resonator. This resonator consists of one chamber and one duct. A chamber acts as air spring and a duct acts as mass. This is simple vibration problem. Fig. 2 shows cavity resonator and its equivalent model.

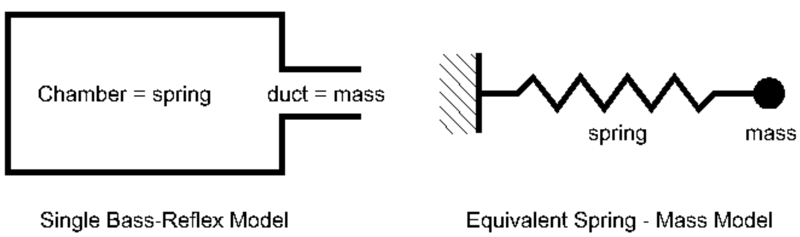

Fig.2 Physical Model of Single Bass-Reflex System

Equations of cavity resonator are derived using state equation of ideal gas. It is expressed in equation (1) under adiabatic condition.

$$PV^{\gamma}=\text{constant} \tag{1}$$

Where,

P: Absolute air pressure inside the chamber [Pa]

V : Capacity of chamber[m³]

γ: Ratio of Specific Heats ($\gamma=1.40$ for air).

Total derivative of equation (1) is expressed as:

$$d(PV^{\gamma})=V^{\gamma}dP+P\cdot\gamma V^{\gamma-1}dV=0 \quad .$$

Multiplying both terms of above equation by $V^{-(\gamma-1)}$, we get:

$$VdP+\gamma PdV=0$$

$$dP=-\frac{\gamma P}{V}dV \tag{2}$$

Hooke's law is expressed as equation (3)

$$dF=-kdx \tag{3}$$

where,

k: Spring constant of the chamber for the mass[N/m]

F: Force acting to mass[N].

Now,

$$\begin{cases} dF=a\cdot dP \\ dV=a\cdot dx \end{cases} \tag{4}$$

where,

a: Cross sectional area of the duct[m³].

Substituting equation (4) for equation (3), we get

$$dF=-\frac{a\gamma P}{V}dV=-\frac{a^2\gamma P}{V}dx \quad .$$

Therefore, spring constant of the chamber is expressed as

$$k=\frac{\gamma a^2 P}{V} \quad (5).$$

By the way, spring constant of the chamber under isothermal condition is expressed as

$$k=\frac{a^2 P}{V} \quad (5)'.$$

Above equations are the basic equations of single bass-reflex system.

Characteristic frequency of the system under adiabatic condition is expressed by equation (6).

$$f_D=\frac{1}{2\pi}\sqrt{\frac{k}{m}}=\frac{1}{2\pi}\sqrt{\frac{\gamma a P}{\rho l V}} \quad [\text{Hz}] \quad (6)$$

In the same manner, characteristic frequency of the system under isothermal condition is expressed by equation (6)'.

$$f_D=\frac{1}{2\pi}\sqrt{\frac{a P}{\rho l V}} \quad [\text{Hz}] \quad (6)'$$

where,

$$m=\rho a l$$

m: Mass of air in the duct[kg]

ρ: Density of air[kg/m^3]

l : Equivalent length of duct[m]

These are the basic equations of single bass-reflex system. These are also used to derive equations of MCAP-CR.

<u>Equation of Motion of MCAP-CR of Free Vibration</u>

In general, adiabatic condition should be used; however, I applied isothermal condition for MCAP-CR calculation, because isothermal condition made better results than adiabatic condition. Please note this assumption is different from public understanding.

Fig. 3 defines axis of each motion where N=4. Arrows stand for positive direction of displacement (x_1 - x_8). k_o - k_8 stands for spring constants of chambers for reference cross sectional area. Equivalent area of speaker corn is used as reference value of cross sectional area for simplicity.

Following naming rule of parameters is applied in this calculation.

A) Subscript 0 references parameter of main chamber.

B) Subscripts 1 through N reference parameters of sub-chambers.

C) Subscripts N+1 through 2N reference for parameters of open-to-air ducts.

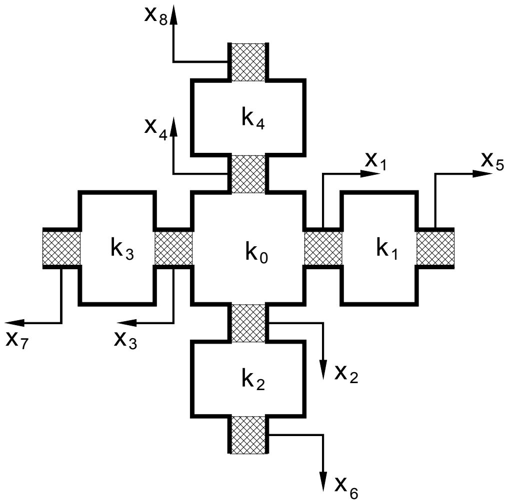

Fig.3 Definition of Directions of Parameters of MCAP-CR (N=4)

Spring constant of each chamber *k* is a function of cross sectional area of duct, so we use factor of each cross sectional area of duct divided by reference area. Reference area a_0 is defined as effective corn area of speaker unit.

Let us define $V_1, V_{2,\ldots}, V_N$ as capacity of each chamber, then we get spring constant of each chamber for reference area a_0 as expressed in equation (7).

$$k_j = \frac{a_0^2 P}{V_j} \tag{7}$$

Let us define cross sectional area of each duct as $a_1, a_2, \ldots, a_N, a_{N+1}, \ldots, a_{2N}$, then spring constant of each chamber for each duct is expressed in equation (8). Subscripts 1, 2, ..., N reference factors of ducts between main chamber and sub-chambers, and subscripts N+1, N+2, ..., 2N reference factors of open-to-air ducts.

$$k_j^* = \frac{r_j^2 a_0^2 P}{V_j} \tag{8}$$

where,

$$r_j = \frac{a_j}{a_0} \quad .$$

Equations of motion of free vibration of this system are expressed in equations (9), where x_j represents displacement of each mass.

$$\begin{cases} m_j \dfrac{d^2 x_j}{dt^2} + k_0 r_j \sum_{i=1}^{N} r_i x_i + k_j r_j (r_j x_j - r_{j+N} x_{j+N}) = 0 \\ m_{j+N} \dfrac{d^2 x_{j+N}}{dt^2} + k_j r_j (r_{j+N} x_{j+N} - r_j x_j) = 0 \end{cases} \qquad (9)$$

where,

j: j=1, 2, ..., N

N: Number of sub-chambers.

An example of 8 degrees of freedom case (N=4) is shown below. Equation of motion in matrix format is expressed in equation (10).

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = 0 \qquad (10)$$

where,

$$\boldsymbol{M} = \begin{bmatrix} m_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & m_7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & m_8 \end{bmatrix} \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} \quad \ddot{\boldsymbol{x}} = \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \\ \ddot{x}_5 \\ \ddot{x}_6 \\ \ddot{x}_7 \\ \ddot{x}_8 \end{bmatrix}$$

$$\boldsymbol{K} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} \quad \boldsymbol{A} = \begin{bmatrix} (k_0+k_1) r_1^2 & k_0 r_1 r_2 & k_0 r_1 r_3 & k_0 r_1 r_4 \\ k_0 r_2 r_1 & (k_0+k_2) r_2^2 & k_0 r_2 r_3 & k_0 r_2 r_4 \\ k_0 r_3 r_1 & k_0 r_3 r_2 & (k_0+k_3) r_3^2 & k_0 r_3 r_4 \\ k_0 r_4 r_1 & k_0 r_4 r_2 & k_0 r_4 r_3 & (k_0+k_4) r_4^2 \end{bmatrix}$$

$$\boldsymbol{B} = \boldsymbol{C} = \begin{bmatrix} -k_1 r_1 r_5 & 0 & 0 & 0 \\ 0 & -k_2 r_2 r_6 & 0 & 0 \\ 0 & 0 & -k_3 r_3 r_7 & 0 \\ 0 & 0 & 0 & -k_4 r_4 r_8 \end{bmatrix}$$

$$\boldsymbol{D} = \begin{bmatrix} k_1 r_5^2 & 0 & 0 & 0 \\ 0 & k_2 r_6^2 & 0 & 0 \\ 0 & 0 & k_3 r_7^2 & 0 \\ 0 & 0 & 0 & k_4 r_8^2 \end{bmatrix} .$$

m_j stands for mass of involved air in each duct, i.e. $m_j = \rho a_j l_j$

where, ρ and l_j stand for respectively density of air and effective length of each duct.

4. Solution of Characteristic Frequencies of MCAP-CR

We need calculation to solve equation (9), so let us calculate eigen values of the equation of motion (10) in matrix format. Eigen values of the equation of motion can be calculated to solve equation (11).

$$|\boldsymbol{K}-\lambda\boldsymbol{M}|=0 \tag{11}$$

or

$$|\boldsymbol{M}^{-1}\boldsymbol{K}-\lambda\boldsymbol{E}|=0 \qquad (11)'.$$

Since equation (11) configures a polynomial whose degree is 2N, then we could get 2N roots. Solution may include multiple root and/or complex roots.
Characteristic frequencies can be calculated after getting eigen values of this problem as follows.

$$f_k=\frac{\sqrt{\lambda_k}}{2\pi}=\frac{\omega_k}{2\pi} \tag{12}$$

where,

λ_k : Eigen values
ω_k : Angular frequency [rad/s]
f_k : Frequency [Hz]
k = 1, 2, ..., 2N.

We have got all the procedure to solve characteristic frequencies of MCAP-CR as shown above.
Since it is very tough to solve eigen value problems of degree of three or greater, we should use computer programs to calculate numerically. There are a number of algorithms to calculate eigen values. The best way to solve MCAP-CR equations is to use proven commercial programs; however, it may cost a lot to purchase one.

Relatively simple codes are to use frame algorithm or Jacob's algorithm; however, there remain some problems. Frame algorithm is not suitable to calculate numerically, because calculation errors may not be negligible. I tried to use frame algorithm, but I saw huge error and its result was not practical. I also tried to use Jacob's algorithm. It worked fine, but I had to design the MCAP-CR to make same mass of air involved in each duct.

Practically the simplest solution was to calculate determinant values numerically as a function of frequency every discrete frequencies over supposed frequency range. Characteristic frequency exists in a range between two consecutive discrete frequencies where sign changes negative to positive or positive to negative. This resolution of calculation should be practically enough, because there are many more uncertain factors.

Appendix -A

In case N=1 (Double Bass-Reflex System)

MCAP-CR requires N=2 or greater, but same equations can be used where N=1. This calculation is simpler and can analytically solve without computer programs. Equation (9) is simplified to equation (A1).

$$\begin{cases} m_1 \ddot{x}_1 + (k_0 + k_1) r_1 x_1 - k_1 r_1 r_2 x_2 = 0 \\ m_2 \ddot{x}_2 - k_1 r_1 r_2 x_1 + k_1 r_2^2 x_2 = 0 \end{cases} \quad \text{(A1)}$$

Expressing equation (A1) in matrix format, then (A1) becomes equation (A2) or (A3).

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{KX} = \boldsymbol{0} \quad \text{(A2)}$$

$$\ddot{\boldsymbol{x}} + \boldsymbol{M}^{-1}\boldsymbol{K}\boldsymbol{x} = 0 \quad \text{(A3)}$$

Characteristic equation of (A3) is as follows:

$$\left|\boldsymbol{M}^{-1}\boldsymbol{K} - \lambda \boldsymbol{E}\right| = 0 \quad \text{(A4)}$$

where,

$$\boldsymbol{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad \boldsymbol{K} = \begin{bmatrix} (k_0 + k_1) r_1^2 & -k_1 r_1 r_2 \\ -k_1 r_2 r_1 & k_1 r_2^2 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \boldsymbol{E} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let us calculate an example of DB-3[2] by Nagaoka.

Principal dimensions of DB-3 are given in Table 1A and1B.

Table1 A   Main Chamber

	W[m]	D[m]	H[m]	Capacity[m^3]	Capacity [l]
Chamber	0.224	0.194	0.155	0.006736	6.74
Duct	0.070	0.070	0.111	-0.00054	-0.54
Displacement of speaker unit				-0.00010	-0.1
Total				**0.006092**	**6.09**

Table 1B   Sub-Chamber

	W[m]	D[m]	H[m]	Capacity[m^3]	Capacity [l]
Chamber	0.224	0.194	0.480	0.02086	20.86
Duct	0.075	0.075	0.060	-0.00034	-0.34
Ribs	0.224	0.015	0.036	-0.00012	-0.12
Total				**0.020400**	**20.40**

2 Tetsuo Nagaoka, " Newest original Speaker Craft 20 (Japanese)", pp113-116, Ongakunotomo (1986)

Equation (A4) is transformed as equation (A5).

$$(k_{11}-\lambda m_1)(k_{22}-\lambda m_2)-k_{12}k_{21}=0 \qquad \text{(A5)}$$

Arranging equation (A5) to polynomial form, we get

$$m_1 m_2 \lambda^2-(k_{11}m_2-k_{22}m_1)\lambda-k_{12}k_{21}+k_{11}k_{12}=0 \qquad \text{(A6).}$$

Because it is obvious that neither $m_1 = 0$ nor $m_2 = 0$, formula of root is applicable, then we get

$$\lambda=\frac{k_{11}m_2+k_{22}m_1\pm\sqrt{(k_{aa}m_2+k_{22}m_1)^2-4m_1m_2(k_{11}k_{22}-k_{12}k_{21})}}{2m_1m_2} \qquad \text{(A7).}$$

Characteristic frequencies are then calculated as

$$f_1=\frac{\sqrt{\lambda_1}}{2\pi} \quad , \quad f_2=\frac{\sqrt{\lambda_2}}{2\pi}$$

Calculation results were f_1=91.3Hz and 45.2Hz.
Reference by Nagaoka gives approximate formulae. Using the reference formulae, we get fd1=94.0Hz and fd2=51.5Hz. There results are practically close enough.
Please note that my equations does not assume adiabatic but isothermal condition as I already discussed. In any case my equations are practical enough for double bass-reflex systems.

Appendix-B

In case N≧2 (4 degrees of freedom or greater): True MCAP-CR

In Appendix-A we have already solved the simplest case (N=1: double bass-reflex); however, it is necessary to use numerical calculation to solve general cases (N≧2). We know that it is not simple to calculate eigen values of three or more degrees of freedom problems.

We will try the simplest method to estimate eigen values of multiple degree of freedom free vibration problems.

We know that eigen values are roots of equation (11). We consider the function (B1).

$$G(\lambda)=\left|\boldsymbol{M}^{-1}\boldsymbol{K}-\lambda\boldsymbol{E}\right| \quad \text{(B1)}$$

where,

$$\lambda=\omega^2=4\pi^2 f^2 \quad \text{(B2)}$$

f : Frequency [Hz]

ω : Angular frequency [rad/sec]

We consider $g(\lambda)$ in order to reduce number of digit of the value, because $G(\lambda)$ will become very big value. Calculating $G(\lambda)$ will increase numerical error.

$$g(\lambda)=\frac{m_1}{k_{11}}\cdot\frac{m_2}{k_{22}}\cdots\frac{m_L}{k_{\mathrm{LL}}}\cdot G(\lambda)=\frac{m_1}{k_{11}}\cdot\frac{m_2}{k_{22}}\cdots\frac{m_L}{k_{\mathrm{LL}}}\cdot\left|\boldsymbol{M}^{-1}\boldsymbol{K}-\lambda\boldsymbol{E}\right| \quad \text{(B3)}$$

where,

L = 2N.

Let us define discrete value of frequency as

$$f_k=f_{\mathrm{min}}+\Delta f \quad \text{(B4).}$$

then,

$$\lambda_k=4\pi^2 f_k^2 \quad \text{(B5).}$$

f_{min} shall be defined by ourselves. It must be 0 or greater. Δf shall also by defined by ourselves. It depends on which resolution we need. Δf =1 [Hz] will be practical enough.

We calculate the following function one by one.

$$g(\lambda)=\frac{m_1}{k_{11}}\cdot\frac{m_2}{k_{22}}\cdots\frac{m_L}{k_{\mathrm{LL}}}\cdot\left|\boldsymbol{M}^{-1}\boldsymbol{K}-4\pi^2 f_k\boldsymbol{E}\right| \quad \text{(B6)}$$

This calculation will let us know where the roots are in resolution range.

Fig. B-1 shows a result of a case N=3.

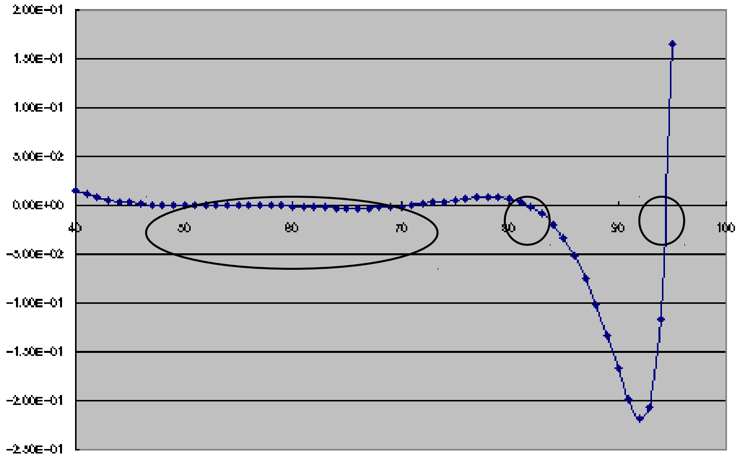

Fig. B-1　Example of Calculation Result (N=3)

Horizontal axis is defines as frequency and vertical axis is defined as $g(f_k)$. Characteristic frequencies are in surrounded area by ellipses where the curve crosses horizontal axis.

Characteristic frequencies in two ellipses in the right side seems clear where they are, but region in the left ellipse does not give clear view to us. Expanded view of this region is seen in Fig. B-2. Fig. B-2 gives clearer view so that we know where other characteristic frequencies are.

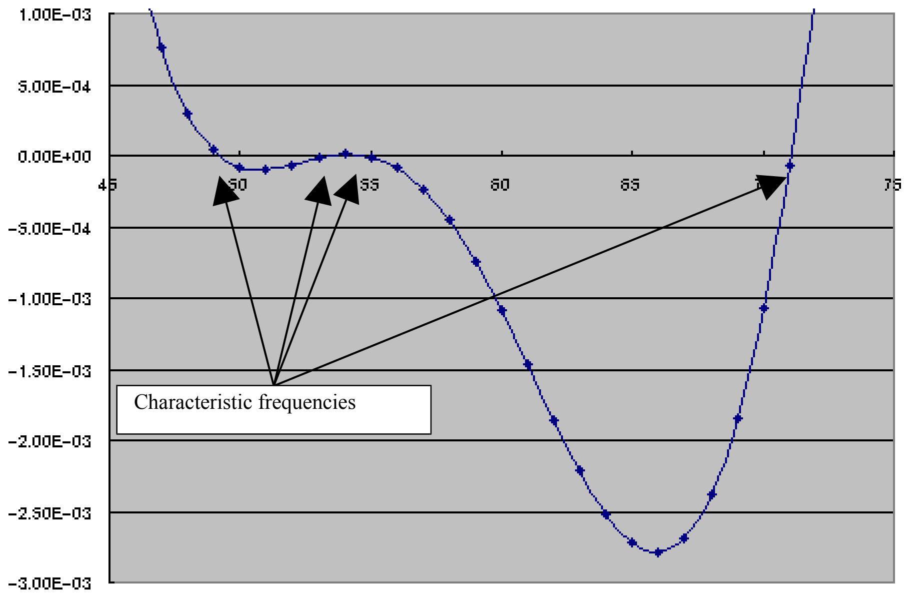

Fig. B-2 Partially Expanded View of Fig. B-1

$g(f_k)$ should be expressed in the polynomial form of 6 degrees; however, these plots seems a little bit different. Extended research on this will be necessary; however, the curve looks nice.

Frame algorithm gave me coefficients of the polynomial, but it did not give me reasonable solution, because coefficients became too big and round off error was not negligible.

In any way, method that was given here will be practical enough to estimate MCAP-CR characters.